# HE 215 : Nuclear & Particle Physics Course

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# Outline I

- The Feynman Calculus
  - Decays and Scattering
  - Fermi's Golden Rule
  - Fermi's Golden Rule for Decays
  - Golden Rule for Scattering
  - The Feynman Rules for a Toy Theory

# The Feynman Calculus

This is chapter 6 in Griffiths.

# The Feynman Calculus

We are now going to skip (for now, will cover it later) chapter 5 on "Bound States" and go straight to chapter 6 of Griffiths.

Chapter 6 introduces the Feynman Calculus with an excellent toy theory before we apply the methods to QED, QCD or electroweak theory.

# Decays and Scattering

- We have three experimental probes of elementary particle interactions:
  - bound states (covered later)
  - decays
  - scattering
- Nonrelativistic quantum mechanics (in Schrodinger's formulation) is particularly well adapted to handle bound states
- By contrast, the relativistic theory (in Feynman's formulation) is especially well suited to describe decays and scattering.

# Observables

We want to relate experimental measurements to theoretical predictions.

- Decay widths and lifetimes
  $\Gamma = h/\tau$ (units of energy)
- Scattering cross-sections
  $\sigma$ is the total cross section
  $\frac{d\sigma}{d\Omega}$ is the angular distribution
  $\frac{d\sigma}{dE}$ is the energy distribution

# Decay Rates

The decay rate $\Gamma$ is the probability per unit time that any given particle will disintegrate. So, for a group of particles we have

$$dN = -\Gamma N dt$$

$$N(t) = N(0)e^{-\Gamma t}$$

The mean lifetime is defined as the reciprocal of the decay rate

$$\tau = \frac{1}{\Gamma}$$

Most particles can decay via multiple distinct channels and the total decay rate is the sum of the rate in each channel

$$\Gamma_{tot} = \sum_{i=1}^{n} \Gamma_i$$

# Breit-Wigner Resonance

Wavefunction for a particle with rest energy $E_R$ and width $\Gamma$ is

$\Psi(t) = \Psi(0)e^{-i(E_R - i\Gamma/2)t}$

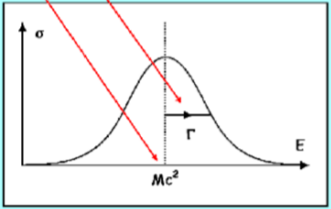
Fourier transform $\quad \chi(E) \propto \frac{1}{(E_R - E) - i\Gamma/2}$

Breit-Wigner formula : $\quad |\chi(E)|^2 \propto \frac{1}{(E_R - E)^2 + \Gamma^2/4}$

The production cross section (rate of production per incoming particle) is described by the Breit Wigner resonance formula

$$\sigma(E) \sim \frac{\Gamma^2}{(E - Mc^2)^2 + \Gamma^2/4}$$

where M is the central mass of the particle and Γ is its width.



# Decay Rates

The Branching Ratio is the fraction of all decaying particles which decay via a specific channel and is logically defined as

$$BR_i = \frac{\Gamma_i}{\Gamma_{tot}}$$

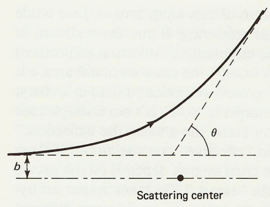
OK, those are the important terms for decays, how about scattering? Well, ultimately we are interested in the cross section...but need to develop some formalism to get there...

# Scattering

A cross section is a geometric idea you can relate to from macroscopic collisions. We need to generalize it.

1. The interaction between the projectile and the target can be "long-range".
2. The cross section is not a sole property of the target but rather a joint characteristic of the projectile and the target.
3. There are inelastic processes in which the final state particles are different from the initial state particles.

Just using classical mechanics we can describe scattering as a way of relating an impact parameter to a scattering angle.



# Hard-Sphere Scattering

## Example 6.1

A particle bounces elastically off of a sphere of radius $R$.

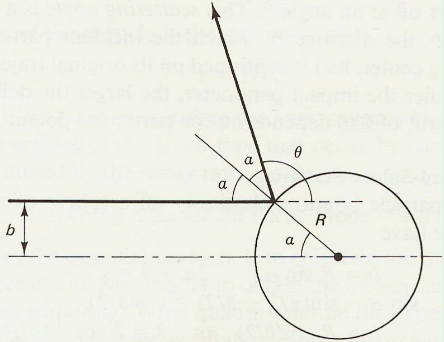


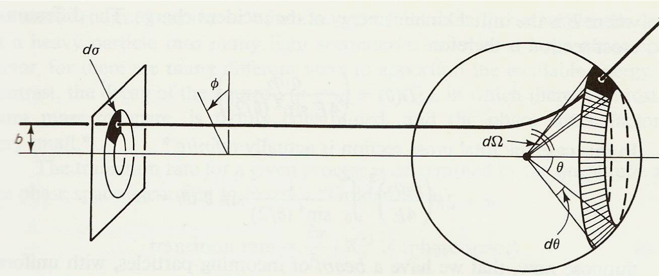

**Figure 6.2** Hard-sphere scattering.

# Hard-Sphere Scattering

$$
\begin{aligned}
b &= R\sin\alpha, \\
2\alpha + \theta &= \pi \\
\sin\alpha &= \sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right) = \cos\left(\frac{\theta}{2}\right) \\
\Longrightarrow\ b &= R\cos\left(\frac{\theta}{2}\right)
\end{aligned}
$$

This is the relation between $\theta$ and $b$ for classical hard-sphere scattering.

# The Differential Cross Section




If a particle passes through an infinitessimal area $d\sigma$ it will be deflected into a corresponding solid angle $d\Omega$. The larger we make $d\sigma$ the larger is $d\Omega$ and the differential cross section is

$$\begin{aligned} d\sigma &= |b\, db\, d\phi|, d\Omega = |\sin\theta\, d\theta\, d\phi| \\ \frac{d\sigma}{d\Omega} &= \left|\frac{b}{\sin\theta}\left(\frac{db}{d\theta}\right)\right| \end{aligned}$$

# Short-range interaction: Hard-Sphere Scattering

## Examples 6.2 and 6.3

Find the differential and total cross sections for hard sphere scattering with a sphere of radius $R$.

From example 6.1 we know that $b = R\cos(\theta/2)$ so

$$\frac{db}{d\theta} = -\frac{R}{2}\sin\left(\frac{\theta}{2}\right)$$

and

$$\frac{d\sigma}{d\Omega} = \frac{Rb\sin(\theta/2)}{2\sin\theta} = \frac{R^2}{2}\frac{\cos(\theta/2)\sin(\theta/2)}{\sin\theta} = \frac{R^2}{4}$$

Giving a total cross section of

$$\sigma = \int \frac{d\sigma}{d\Omega}d\Omega = \int \frac{R^2}{4}d\Omega = \pi R^2$$

# Long-range interaction: Rutherford Scattering

## Example 6.4 - Rutherford Scattering

A particle of charge $q_1$ scatters off a stationary particle of charge $q_2$.

In classical mechanics the formula relating $b$ to $\theta$ is

$$b = \frac{q_1 q_2}{2E} \cot(\theta/2)$$

So, the differential cross section is

$$\frac{d\sigma}{d\Omega} = \left| \frac{b}{\sin\theta} \left( \frac{db}{d\theta} \right) \right| = \left( \frac{q_1 q_2}{4E \sin^2(\theta/2)} \right)^2$$

The total cross section is actually infinite...
This is related to the fact that the Coulomb potential has infinite range..

# Fermi's Golden Rule

**Fermi's Golden Rule**

$$\textit{transition rate} = \frac{2\pi}{\hbar}|\mathcal{M}|^2 \times (\textit{phase space})$$

- The amplitude $\mathcal{M}$ contains all of the dynamical information. Use Feynman diagram/rules to calculate this.
- The phase space is a kinematic factor. The bigger the phase space the larger the transition rate.
- Alternate terminology:
  - Amplitude $\leftrightarrow$ Matrix Element
  - Phase Space $\leftrightarrow$ Density of Final States

# Fermi's Golden Rule for Decays

## Golden Rule for Decays

Suppose particle 1 decays as

$$1 \rightarrow 2 + 3 + 4 + \cdots + n$$

The decay rate is given by

$$\Gamma = \frac{S}{2\hbar m_1} \int |\mathcal{M}|^2 (2\pi)^4 \delta^4 \left(p_1 - p_2 - p_3 \cdots - p_n\right) \times \prod_{j=2}^{n} 2\pi \delta \left(p_j^2 - m_j^2 c^2\right) \theta \left(p_j^0\right) \frac{\mathrm{d}^4 p_j}{(2\pi)^4}$$

Where the $\delta$ is a Dirac-Delta, $\theta$ is step function, $p_i$ is the 4-momentum of the $i$-th particle, the decaying particle is at rest $(m_1 c, \mathbf{0})$ and $S$ is $1/j!$ for each group of $j$ identical particles in the final state.

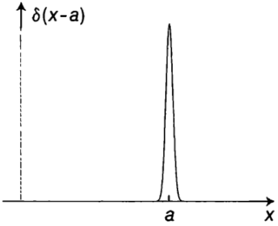
# Dirac Delta & $\theta$ functions

## Dirac delta function

$$\delta(x-a) = \begin{cases} 0, & \text{if } x \neq a \\ \infty, & \text{if } x = a \end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(x-a)\,dx = 1$$

$$f(x)\delta(x-a) = f(a)\delta(x-a)$$

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)\,dx = f(a)$$




## Step function

$$\theta(x) \equiv \begin{cases} 0, & (x < 0) \\ 1, & (x > 0) \end{cases}$$

# Fermi's Golden Rule for Decays

The dynamics of the process is contained in amplitude, $\mathcal{M}$, which we will calculate later using appropriate Feynman diagrams applying Feynman rules.
The rest is phase space which tells us to integrate over all outgoing four-momenta, subject to three kinematical constraints:

1. Each outgoing particle lies on its mass shell: $p_j^2 = m_j^2 c^2$ (which is to say, $E_j^2 - \mathbf{p}_j^2 c^2 = m_j^2 c^4$). This is enforced by the delta function $\delta\left(p_j^2 - m_j^2 c^2\right)$, which is zero unless its argument vanishes.
2. Each outgoing energy is positive: $p_j^0 = E_j/c > 0$. Hence the $\theta$ function.
3. Energy and momentum must be conserved: $p_1 = p_2 + p_3 \cdots + p_n$. This is ensured by the factor $\delta^4(p_1 - p_2 - p_3 \cdots - p_n)$.

# Fermi's Golden Rule for Decays

- Thumb rule for factors of $2\pi$:
  Every $\delta$ gets $(2\pi)$ and d gets $1/2\pi$.
- Four-dimensional 'volume' elements can be split into spatial and temporal parts: $d^4p = dp^0 d^3\boldsymbol{p}$
- Using the properties of $\delta$ function:

$$\delta\left(p^2 - m^2c^2\right) = \delta\left[(p^0)^2 - \mathbf{p}^2 - m^2c^2\right]$$

$$\delta(x^2 - a^2) = \frac{1}{2a}\left[\delta(x-a) + \delta(x+a)\right] \quad (a > 0)$$

$$\theta\left(p^0\right)\delta\left[(p^0)^2 - \mathbf{p}^2 - m^2c^2\right] = \frac{1}{2\sqrt{\mathbf{p}^2 + m^2c^2}}\delta\left(p^0 - \sqrt{\mathbf{p}^2 + m^2c^2}\right)$$

$\theta$ function kills the spike at $p^0 = -\sqrt{\mathbf{p}^2 + m^2c^2}$

# Fermi's Golden Rule for Decays

Thus the total decay rate is given by:

$$\Gamma = \frac{S}{2\hbar m_1} \int |\mathcal{M}|^2 (2\pi)^4 \delta^4 \left(p_1 - p_2 - p_3 \cdots - p_n\right)$$

$$\times \prod_{j=2}^{n} \frac{1}{2\sqrt{\mathbf{p}_j^2 + m_j^2 c^2}} \frac{\mathrm{d}^3 \mathbf{p}_j}{(2\pi)^3}$$

with

$$p_j^0 \to \sqrt{\mathbf{p}_j^2 + m_j^2 c^2}$$

# Two-particle Decays

With only 2 particles in the final state we have:

$$\Gamma = \frac{S}{32\pi^2\hbar m_1}\int |\mathcal{M}|^2 \frac{\delta^4(p_1 - p_2 - p_3)}{\sqrt{\mathbf{p}_2^2 + m_2^2c^2}\sqrt{\mathbf{p}_3^2 + m_3^2c^2}}\, \mathrm{d}^3\mathbf{p}_2 \times \mathrm{d}^3\mathbf{p}_3$$

The four-dimensional delta function is a product of temporal and spatial parts:

$$\delta^4(p_1 - p_2 - p_3) = \delta\left(p_1^0 - p_2^0 - p_3^0\right)\delta^3\left(\mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3\right)$$

With particle 1 at rest, the decay rate is:

$$\Gamma = \frac{S}{32\pi^2\hbar m_1}\int |\mathcal{M}|^2 \frac{\delta\left(m_1c - \sqrt{\mathbf{p}_2^2 + m_2^2c^2} - \sqrt{\mathbf{p}_3^2 + m_3^2c^2}\right)}{\sqrt{\mathbf{p}_2^2 + m_2^2c^2}\sqrt{\mathbf{p}_3^2 + m_3^2c^2}}$$
$$\times\, \delta^3\left(\mathbf{p}_2 + \mathbf{p}_3\right)\, \mathrm{d}^3\mathbf{p}_2\, \mathrm{d}^3\mathbf{p}_3$$

# Two-particle Decays

$p_3$ integral is now trivial: in view of the final delta function replace $p_3 \rightarrow -p_2$

$$\Gamma = \frac{S}{32\pi^2 \hbar m_1} \int |\mathscr{M}|^2 \frac{\delta\left(m_1 c - \sqrt{\mathbf{p}_2^2 + m_2^2 c^2} - \sqrt{\mathbf{p}_2^2 + m_3^2 c^2}\right)}{\sqrt{\mathbf{p}_2^2 + m_2^2 c^2}\sqrt{\mathbf{p}_2^2 + m_3^2 c^2}} \, \mathrm{d}^3\mathbf{p}_2$$

Using spherical co-ordinates $p_2 \rightarrow (r, \theta, \phi)$ and $d^3 p_2 \rightarrow r^2 sin\theta dr d\theta d\phi$ with $r = |p_2|$:

$$\Gamma = \frac{S}{32\pi^2 \hbar m_1} \int |\mathscr{M}|^2 \frac{\delta\left(m_1 c - \sqrt{r^2 + m_2^2 c^2} - \sqrt{r^2 + m_3^2 c^2}\right)}{\sqrt{r^2 + m_2^2 c^2}\sqrt{r^2 + m_3^2 c^2}}$$

$$\times\, r^2 \sin\theta \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\phi$$

# Two-particle Decays

$\mathcal{M}$ is only function of $r$ and integrating over the angles $\int \sin\theta \, d\theta \, d\phi = 4\pi$ gives

$$\Gamma = \frac{S}{8\pi\hbar m_1} \int_0^\infty |\mathcal{M}(r)|^2 \frac{\delta\left(m_1 c - \sqrt{r^2 + m_2^2 c^2} - \sqrt{r^2 + m_3^2 c^2}\right)}{\sqrt{r^2 + m_2^2 c^2}\sqrt{r^2 + m_3^2 c^2}} r^2 \, dr$$

Let

$$u \equiv \sqrt{r^2 + m_2^2 c^2} + \sqrt{r^2 + m_3^2 c^2}$$

so

$$\frac{du}{dr} = \frac{ur}{\sqrt{r^2 + m_2^2 c^2}\sqrt{r^2 + m_3^2 c^2}}$$

# Two-particle Decays

$$\Gamma = \frac{S}{8\pi\hbar m_1}\int_{(m_2+m_3)c}^{\infty} |\mathcal{M}(r)|^2 \delta\left(m_1 c - u\right) \frac{r}{u}\, du$$

Last integral sends $u$ to $m_1 c$ and hence $r$ to:

$$r_0 = \frac{c}{2m_1}\sqrt{m_1^4 + m_2^4 + m_3^4 - 2m_1^2 m_2^2 - 2m_1^2 m_3^2 - 2m_2^2 m_3^2}$$

# Two-particle Decays

$$m_1 c = \sqrt{r^2 + m_2^2 c^2} + \sqrt{r^2 + m_3^2 c^2}. \quad \text{Square:}$$

$$m_1^2 c^2 = r^2 + m_2^2 c^2 + r^2 + m_3^2 c^2 + 2\sqrt{r^2 + m_2^2 c^2}\sqrt{r^2 + m_3^2 c^2}$$

$$\frac{c^2}{2}(m_1^2 - m_2^2 - m_3^2) - r^2 = \sqrt{r^2 + m_2^2 c^2}\sqrt{r^2 + m_3^2 c^2}. \quad \text{Square again:}$$

$$\frac{c^4}{4}(m_1^2 - m_2^2 - m_3^2)^2 - r^2 c^2 (m_1^2 - m_2^2 - m_3^2) + \cancel{r^4} = \cancel{r^4} + r^2 c^2 (m_2^2 + m_3^2) + m_2^2 m_3^2 c^4$$

$$\frac{c^4}{4}\left[(m_1^2 - m_2^2 - m_3^2)^2 - 4m_2^2 m_3^2\right] = r^2 c^2 (m_2^2 + m_3^2 + m_1^2 - m_2^2 - m_3^2) = r^2 m_1^2 c^2$$

$$r^2 = \frac{c^2}{4m_1^2}\left[m_1^4 + m_2^4 + m_3^4 - 2m_1^2 m_2^2 - 2m_1^2 m_3^2 + 2m_2^2 m_3^2 - 4m_2^2 m_3^2\right]$$

$$\boxed{r = \frac{c}{2m_1}\sqrt{m_1^4 + m_2^4 + m_3^4 - 2m_1^2 m_2^2 - 2m_1^2 m_3^2 - 2m_2^2 m_3^2}} \quad \checkmark$$

# Two-particle Decays

$$\Gamma = \frac{S}{8\pi\hbar m_1} \int_{(m_2+m_3)c}^{\infty} |\mathcal{M}(r)|^2 \delta(m_1 c - u) \frac{r}{u} \, du$$

Last integral sends $u$ to $m_1 c$ and hence $r$ to:

$$r_0 = \frac{c}{2m_1}\sqrt{m_1^4 + m_2^4 + m_3^4 - 2m_1^2 m_2^2 - 2m_1^2 m_3^2 - 2m_2^2 m_3^2}$$

$r_0$ is particular value of $|p_2|$ that is consistent with conservation of energy. In comprehensive notation:

$$\Gamma = \frac{S|\mathbf{p}|}{8\pi\hbar m_1^2 c}|\mathcal{M}|^2$$

where $|p|$ is magnitude of outgoing momentum, given in terms of three masses (eqn of $r_0$).

# Two-particle Decays

For massive particles we have the very useful expression:

$$\Gamma = \frac{S|\boldsymbol{p}|}{8\pi\hbar m_1^2 c}|\mathcal{M}|^2$$

Example decay: $\rho \rightarrow \pi + \pi$

For massless particles using conservation of energy we have $|\boldsymbol{p}| = (m_1 c)/2$. Thus decay rate becomes:

$$\Gamma = \frac{S}{16\pi\hbar m_1}|\mathcal{M}|^2$$

Example decay: $\pi \rightarrow \gamma + \gamma$

- In the previous two examples, we have shown how the phase space for the decay rate of a 2-body decay can be integrated completely without any information about $\mathcal{M}$.
- For 3-body decays (and beyond), this is no longer possible, as the amplitude will typically depend non-trivially upon several of the phase space integration variables, so that we have to do the integration by hand for each specific $\mathcal{M}$.

## Golden Rule for Scattering

Suppose particles 1 and 2 collide:

$$1 + 2 \rightarrow 3 + 4 + \cdots + n$$

The cross section is given by

$$\sigma = \frac{S\hbar^2}{4\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2 c^2)^2}} \int |\mathscr{M}|^2 (2\pi)^4 \delta^4(p_1 + p_2 - p_3 \cdots - p_n)$$
$$\times \prod_{j=3}^{n} 2\pi\delta\left(p_j^2 - m_j^2 c^2\right) \theta\left(p_j^0\right) \frac{\mathrm{d}^4 p_j}{(2\pi)^4}$$

Where the variables are all defined as in the decay case.

# Golden Rule for Scattering

The phase space is essentially the same as the decay case: integrate over all outgoing momenta, subject to the three kinematical constraints (**every outgoing particle is on its mass shell, every outgoing energy is positive, and energy and momentum are conserved**), which are enforced by the delta and theta functions.

We can simplify cross section by performing the $p_j^0$ integrals:

$$\sigma = \frac{S\hbar^2}{4\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2 c^2)^2}} \int |\mathcal{M}|^2 (2\pi)^4 \delta^4(p_1 + p_2 - p_3 \cdots - p_n) \times \prod_{j=3}^{n} \frac{1}{2\sqrt{\mathbf{p}_j^2 + m_j^2 c^2}} \frac{d^3\mathbf{p}_j}{(2\pi)^3}$$

with

$$p_j^0 = \sqrt{\mathbf{p}_j^2 + m_j^2 c^2}$$

# 2-body Scattering Example

Consider the process

$$1 + 2 \rightarrow 3 + 4$$

in the CM frame. If the amplitude is $\mathcal{M}$, calculate the differential cross section.


**Fig. 6.5** Two-body scattering in the *CM* frame.

In the CM frame $\boldsymbol{p_2} = -\boldsymbol{p_1}$ giving

$$p_1 \cdot p_2 = \frac{E_1 E_2}{c^2} + \boldsymbol{p}_1^2$$

# 2-body Scattering Example

and you can show (using $E^2 = (pc)^2 + (mc^2)^2$ a few times)

$$\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2 c^2)^2} = (E_1 + E_2)|\boldsymbol{p}_1|/c$$

Which allows us to write the cross section

$$\sigma = \frac{S\hbar^2 c}{64\pi^2 (E_1 + E_2)|\mathbf{p}_1|} \int |\mathscr{M}|^2 \frac{\delta^4(p_1 + p_2 - p_3 - p_4)}{\sqrt{\mathbf{p}_3^2 + m_3^2 c^2}\sqrt{\mathbf{p}_4^2 + m_4^2 c^2}} \, \mathrm{d}^3\mathbf{p}_3 \, \mathrm{d}^3\mathbf{p}_4$$

Delta function can be rewritten as:

$$\delta^4(p_1 + p_2 - p_3 - p_4) = \delta\left(\frac{E_1 + E_2}{c} - p_3^0 - p_4^0\right) \delta^3(\mathbf{p}_3 + \mathbf{p}_4)$$

# 2-body Scattering Example

Now the integral will pick out values corresponding to $\boldsymbol{p}_4 = -\boldsymbol{p}_3$.

$$\sigma = \left(\frac{\hbar}{8\pi}\right)^2 \frac{Sc}{(E_1 + E_2)|\mathbf{p}_1|} \int |\mathscr{M}|^2 \times \frac{\delta\left[(E_1 + E_2)/c - \sqrt{\mathbf{p}_3^2 + m_3^2 c^2} - \sqrt{\mathbf{p}_3^2 + m_4^2 c^2}\right]}{\sqrt{\mathbf{p}_3^2 + m_3^2 c^2}\sqrt{\mathbf{p}_3^2 + m_4^2 c^2}} \, \mathrm{d}^3\mathbf{p}_3$$

Using spherical co-ordinates:

$$d^3\boldsymbol{p}_3 = r^2 dr d\Omega$$

where $r = |\boldsymbol{p}_3|$ and $d\Omega = \sin\theta \, d\theta \, d\phi$

# 2-body Scattering Example

We can not carry angular integration as $\mathcal{M}^2$ depends on direction of $\boldsymbol{p}_3$ and magnitude.
We can thus get differentical cross section as:

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar}{8\pi}\right)^2 \frac{Sc}{(E_1 + E_2)|\mathbf{p}_1|} \int_0^\infty |\mathcal{M}|^2 \times \frac{\delta\left[(E_1 + E_2)/c - \sqrt{r^2 + m_3^2c^2} - \sqrt{r^2 + m_4^2c^2}\right]}{\sqrt{r^2 + m_3^2c^2}\sqrt{r^2 + m_4^2c^2}} r^2\, dr$$

# 2-body Scattering Example

The integral then yields (see the variable substitution as in case of two-body decay)

**Cross Section for 2-body Scattering**

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{S|\mathcal{M}|^2}{(E_1 + E_2)^2} \frac{|\boldsymbol{p}_f|}{|\boldsymbol{p}_i|}$$

where $|\boldsymbol{p}_f|$ is the magnitude of outgoing momentum and $|\boldsymbol{p}_i|$ is magnitude of incoming momentum.

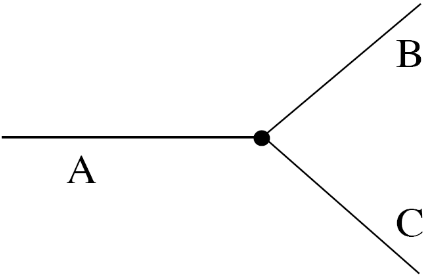
The dimensions of $\mathcal{M}$ are

$$(mc)^{4-n}$$

where $n$ is the number of external lines in the Feynman diagram.

# The Feynman Rules for a Toy Theory

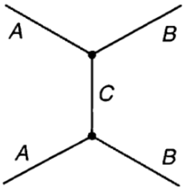
OK, now we know how to take a known amplitude ($\mathcal{M}$) and use it to calculate a decay rate or a cross section....but how do we get $\mathcal{M}$?? We will start with Griffith's Feynman rules for a toy theory. Let's say we live in a world with only 3 kinds of particles ($A, B, C$) and only one vertex:



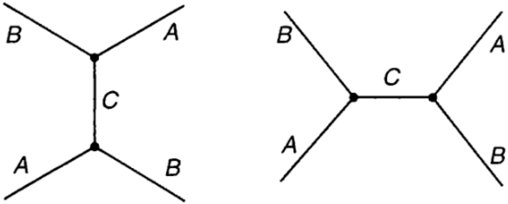

This vertex is also the diagram for $A \rightarrow B + C$ decay (assume $A$ is heavier than $B$ and $C$ combined). These three particles are all spin-0 and each is its own antiparticle.

# The Feynman Rules for a Toy Theory

We want to be able to calculate the lifetime of $A$ to lowest order and calculate the scattering processes $A + A \to B + B$




and $A + B \to A + B$



# The Feynman Rules for a Toy Theory

1. Draw the Feynman diagram(s) for the lowest-order process, ie, with minimum number of vertices. There may be more than one.
2. Label the incoming and outgoing 4-momenta $p_1, p_2, \ldots, p_n$. Label the internal momenta as $q_1, q_2, \ldots, q_n$. Put arrows on each line to indicate the "positive" direction.
3. For each vertex write down a factor of
$$-ig$$
where $g$ is called the coupling constant. For this toy $g$ has dimensions of momentum, normally it is dimensionless.
4. For each internal line, write a factor
$$\frac{i}{q_j^2 - m_j^2 c^2}$$
where $q_j$ is the 4-momentum of the line and $m_j$ is the mass of the particle the line describes.

# The Feynman Rules for a Toy Theory

5. For each vertex, write a delta function of the form

$$(2\pi)^4\delta^4(k_1 + k_2 + k_3)$$

   where the “k”’s are the four-momenta coming **into** the vertex. This imposes conservation of energy and momentum at each vertex.

6. For each internal line, write down a factor

$$\frac{1}{(2\pi)^4}d^4q_j$$

   and integrate over internal momenta

7. You will be able to write the result containing a delta function

$$(2\pi)^4\delta^4(p_1 + p_2 + \cdots - p_n)$$

   erase this factor. Multiply by $i$ in order to get $\mathcal{M}$.

# The Feynman Rules for a Toy Theory

To sum up the Feynman rules:

The amplitude is given by:

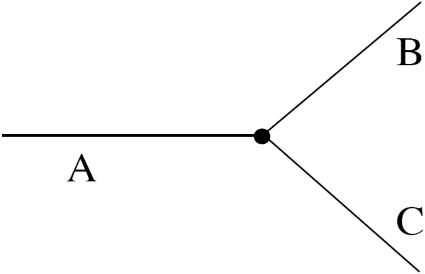
$$\mathcal{M}$$

$$=$$

*i* (vertex factors)
(propagators)
(momentum conservation)
(integration over internal momenta)

# Lifetime of the $A$

The simplest possible diagram is that for the decay of $A$




There are no internal lines, only one vertex so we get

3. vertex coupling: $-ig$
5. conservation of $E$ and $p$: $(2\pi)^4\delta^4(p_A - p_B - p_C)$
7. erase the delta function. Multiply by $i$ in order to get $\mathcal{M}$.

$$\mathcal{M} = g$$

# Lifetime of the $A$

- From Fermi's Golden Rule, the decay rate is given by:

$$\Gamma = \frac{S|\boldsymbol{p}|}{8\pi\hbar m_1^2 c}|\mathcal{M}|^2$$

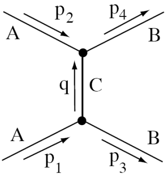
  in the rest frame of A.
- With $S = 1$, $m_1 = m_A$, $\mathcal{M} = g$ and $|\boldsymbol{p}|$ representing the magnitude of the spatial momentum of either $B$ or $C$, we find that

$$\Gamma = \frac{g^2|\boldsymbol{p}|}{8\pi\hbar m_A^2 c}$$

  and the lifetime is $\Gamma^{-1}$

# Scattering in a Toy Model

Consider the lowest order scattering diagram in our toy model




This has two vertices (each $-ig$) and one internal line:

$$\frac{i}{q^2 - m_C^2 c^2}$$

The vertices give 2 delta functions

$$(2\pi)^4 \delta^4(p_1 - p_3 - q), (2\pi)^4 \delta^4(p_2 + q - p_4)$$

and the internal line gives an integration

$$\frac{1}{(2\pi)^4} d^4 q$$

# Scattering in a Toy Model

We then have

$$-i(2\pi)^4 g^2 \int \frac{1}{q^2 - m_C^2 c^2} \delta^4(p_1 - p_3 - q)\delta^4(p_2 + q - p_4) d^4 q$$

The second delta picks out the values at $q = p_4 - p_2$ leaving

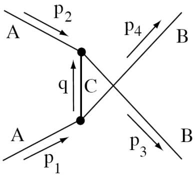
$$-ig^2 \frac{1}{(p_4 - p_2)^2 - m_C^2 c^2} (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4)$$

Remove the remaining delta and we have

$$\mathcal{M} = \frac{g^2}{(p_4 - p_2)^2 - m_C^2 c^2}$$

# Scattering in a Toy Model

But wait! What about:




That has the same initial and final state but a different assignment of final momenta ($p_3 \leftrightarrow p_4$). The total amplitude (at lowest order) for this process is therefore

$$\mathcal{M} = \frac{g^2}{(p_4 - p_2)^2 - m_C^2 c^2} + \frac{g^2}{(p_3 - p_2)^2 - m_C^2 c^2}$$

We add the amplitudes of all diagrams with same same initial and final state before plugging into the cross section expression.

# Higher Order Diagrams

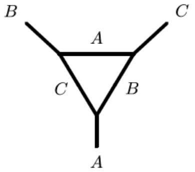
- Of course, we have only looked at the lowest order ("tree level") scattering diagrams. What about diagrams with more vertices?
- These diagrams will generate extra contributions to the amplitude:

$$
\begin{aligned}
\mathcal{M}_{A\to B+C} &= g\mathcal{A}_1 + g^3\mathcal{A}_3 + g^5\mathcal{A}_5 + \cdots \\
\mathcal{M}_{A+B\to C+D} &= g^2\mathcal{A}_2 + g^4\mathcal{A}_4 + g^6\mathcal{A}_6 + \cdots
\end{aligned}
$$

- If $g << 1$ then successive terms in the perturbation series become smaller and smaller corrections to the amplitude.

# Corrections to $A$ Decay

There is one third-order diagram to consider:

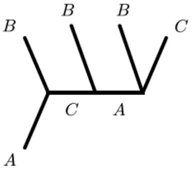



This $\mathcal{O}(g^3)$ contribution to the $\mathcal{O}(g)$ tree-level diagram will come in to the decay rate in a coherent sum:

$$|\mathcal{M}|^2 = |g\mathcal{A}_1 + g^3\mathcal{A}_3|^2$$

# Corrections to $A$ Decay

We can also see diagrams for $A$ decay (assuming $A$ is sufficiently heavy) which do not result in the same final state:



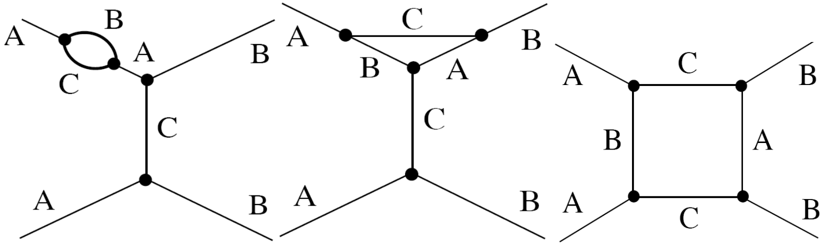

This is $A \to 3B + C$. The total decay rate for $A$ could be something like:

$$\begin{aligned}\Gamma(A \to \text{ anything }) &= \Gamma_{A\to BC} + \Gamma_{A\to BBBC} + \Gamma_{A\to BCCC} + \cdots \\ &= C_1 \left|\sum \mathcal{M}_{A\to BC}\right|^2 + C_2 \left|\sum \mathcal{M}_{A\to BBBC}\right|^2 \\ &\quad + C_3 \left|\sum \mathcal{M}_{A\to BCCC}\right|^2\end{aligned}$$

where the $C$'s are from the Golden Rule. This expression involves both coherent and incoherent sums.

# Scattering Corrections

We also have many possible diagrams to correct scattering. For example:



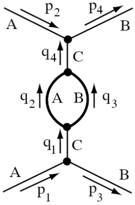

five 'self energy' (left), two 'vertex correction' (center), one 'box' diagram (right)

seven more with external $B$ lines twisted.

In total there are 15 of these fourth-order diagrams. Clearly this could be a lot of work!

# Scattering Corrections

Let's just look at the "vacuum polarization" correction to the amplitude




See pages 218-219 of your text for details, but we end up with

$$\mathcal{M} = i\left(\frac{g}{2\pi}\right)^4 \frac{1}{[(p_1 - p_3)^2 - m_C^2 c^2]^2} \int \frac{d^4 q}{[(p_1 - p_3 - q)^2 - m_A^2 c^2](q^2 - m_B^2 c^2)}$$

Attempting to calculate this integral will give you fits. At large $q$ we have

$$\int^{\infty} \frac{1}{q^4} q^3 dq = \ln q|^{\infty} = \infty$$

# Regularization

- The first step in dealing with loop integral divergences is called regularization. This is an artificial adjustment to the integral so that it can be solved (“sweeping the infinities under the rug”).
- Introduce a cutoff mass. Introduce a factor:

$$\frac{-M^2c^2}{(q^2 - M^2c^2)}$$

- This cutoff mass $M$ is taken to be some very large number. Effectively you change the upper limit of integration from $\infty$ to $M$.
- Some of the infinities introduced in the coherent sum over amplitudes now cancel....but not all. We now have a finite part and an infinite part.
- Now we need renormalization to rescue us.

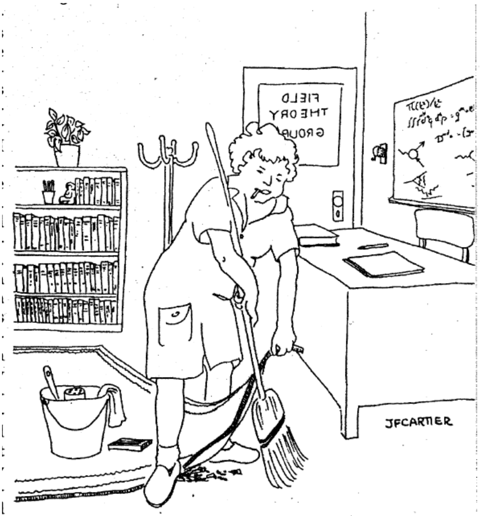
# Renormalization

- All divergences in the final physical observables ($\sigma$,$\Gamma$) appear as additions to the masses and coupling constants.
- So, we can define renormalized masses or couplings which absorb the divergences. We then assume it is the renormalized quantities that we have been observing all along. This means that the bare quantities aren't observable.

$$\begin{aligned} m_{physical} &= m + \delta m \\ g_{physical} &= g + \delta g \end{aligned}$$

- We use the physical masses and couplings in the Feynman rules. (So, I guess we better measure them)
- Once infinities are resolved we still have finite contributions from higher-order diagrams. These are functions of the 4-momentum of the line in which the loop is inserted. So, effective masses and coupling constants depend on energy - they run!
- Renormalizability is a feature of all QFTs.

# Renormalization




ALRIGHT RUTH, I ABOUT GOT THIS ONE RENORMALIZED.